$\dfrac{\partial^2 f}{\partial z^2}$ を求めるには、

$$\frac{\partial \square}{\partial z} = \cos\theta \frac{\partial \square}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \square}{\partial \theta} \qquad \cdots\cdots(1)$$

において、□ に $\dfrac{\partial f}{\partial z}$ を放り込めばよい。すると

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \cos\theta \underline{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right)} - \frac{\sin\theta}{r}\underline{\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial z}\right)} \qquad \cdots\cdots(2)$$

下線部を求めるために再び (1) を用いれば

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) = \cos\theta\frac{\partial^2 f}{\partial r^2} - \sin\theta \underline{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)} \qquad \cdots\cdots(3)$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) = \underline{\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial f}{\partial r}\right)} - \frac{1}{r}\underline{\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right)} \qquad \cdots\cdots(4)$$

ここで、例えば (3) は r についての偏微分であるので、θ は定数であり、$\cos\theta$ や $\sin\theta$ は偏微分の外に出せることに注意。

さらに現れた 3 箇所の下線部を計算すると（「積の微分」を利用する）、

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) = \frac{\partial(1/r)}{\partial r}\frac{\partial f}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial \theta}\right) = -\frac{1}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r}\frac{\partial^2 f}{\partial r \partial \theta} \qquad \cdots\cdots(5)$$

$$\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial f}{\partial r}\right) = \frac{\partial(\cos\theta)}{\partial \theta}\frac{\partial f}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r}\right) = -\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial \theta \partial r} \qquad \cdots\cdots(6)$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) = \frac{\partial(\sin\theta)}{\partial \theta}\frac{\partial f}{\partial \theta} + \sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial \theta}\right) = \cos\theta\frac{\partial f}{\partial \theta} + \sin\theta\frac{\partial^2 f}{\partial \theta^2} \qquad \cdots\cdots(7)$$

(5)(6)(7) を (3)(4) に代入し、さらに (2) に代入すれば

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^2} &= \cos\theta\left[\cos\theta\frac{\partial^2 f}{\partial r^2} - \sin\theta\left(-\frac{1}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r}\frac{\partial^2 f}{\partial r \partial \theta}\right)\right] - \frac{\sin\theta}{r}\left[\left(-\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial \theta \partial r}\right) - \frac{1}{r}\left(\cos\theta\frac{\partial f}{\partial \theta} + \sin\theta\frac{\partial^2 f}{\partial \theta^2}\right)\right] \\
&= \cos\theta\left(\cos\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial f}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial r \partial \theta}\right) - \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial \theta \partial r} - \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial \theta^2}\right) \\
&= \left(\cos^2\theta\frac{\partial^2 f}{\partial r^2} + \underline{\frac{\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial \theta}} - \underline{\underline{\frac{\cos\theta\sin\theta}{r}\frac{\partial^2 f}{\partial r \partial \theta}}}\right) + \left(\frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} - \underline{\underline{\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r}}} + \underline{\frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta}} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right) \\
&= \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \underline{\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial \theta}} - \underline{\underline{\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 f}{\partial r \partial \theta}}} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}
\end{aligned}$$

なお、「普通の」関数では $\dfrac{\partial^2 f}{\partial \theta \partial r} = \dfrac{\partial^2 f}{\partial r \partial \theta}$ であることを用いた。